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On the variational formulation of the extended thick anisotropic shells theory of I. N. Vekua type

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Abstract

A new variant of the I. N. Vekua – A. A. Amosov extended theory of thick anisotropic shells is constructed on the groundwork of the Lagrange variational formalism of analytical mechanics of continua and the dimensional reduction approach. The shell model consists in the set of field variables, the surface Lagrangian density, and the constraint equations defined on the two-dimensional manifold corresponding to the base surface. The supplementary constraints are derived from the boundary conditions on shell's faces. The field variables of the first kind are biorthogonal expansion's coefficients for the displacement vector. This constrained variational problem is solving by the Lagrange multipliers method that results the two-dimensional initial-boundary value problem of the general shell theory.

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Keywords: Thick anisotropic shells; General shell theory; Analytical mechanics of continua; Constrained mechanical systems; Generalized Lagrange equations of the second kind

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1. Introduction

Plates and shells are widely used in civil engineering as well as in machine industry. A lot of shell models are applied to the engineering problems' solution and implemented in finite element software [9], but a refined shell theory is still required for some applications. It is noted that "… more reliable 2D multi-field models are needed for… high-frequency vibrations of shells, … wave propagation etc." [13]. The refinement must consist in accurate modeling of three-dimensional stress state in the "irreducibility domains" [22] near the boundaries of shells, contact areas, or wave fronts. These domains have often the width of the order of shell thickness, therefore their accounting is strongly required in the civil engineering where the shell or plate elements are usually relatively thick [1-3].

Various applications of refined shell theories are discussed in several books recently published [21], and the necessity of high-order modeling is shown. For instance, the effect of normal transverse stress on laminated shells' vibration was investigated in [8]. On the groundwork of several numerical results it was shown that the Koiter's recommendation, "… a refinement of Love's first approximation theories is indeed meaningless… unless the effects of transverse shear and normal stress are taken into account in the same time" [24], is especially urgent for the analysis of layered and/or anisotropic shells' dynamics. Later it was shown that the accurate modeling of high-frequency vibration requires the theory of fourth order if even the considered plate is relatively thin [11]. The solution of type of a boundary layer near plate's faces was found in [6] for the high-frequency oscillations' problem. Kang and al. have shown that the accurate approximated solution for the vibrating conical shell also requires the high-order theory [19]; the authors have used the power series. At the same time Zhou and al. have applied the high-order plate theory using the special basis function to the problem of high-frequency vibration of circular plates [46]. The mechanics of functionally graded plates and shells was investigated by Matsunaga; the statics of plates [26], free vibration of plates [27] and shells [28, 29] were studied in details. Matsunaga has also shown that high-order theories are strongly required for the thermal buckling analysis of functionally graded thick plates [30]. Thus the development of high-order shell theories is still topical for applied mechanics as well as for civil engineering.

A lot of approaches that can be used to construct a high-order shell theory. The asymptotic integration method was proposed by Goldenweiser [14], Cicala [12] and other authors (see [4]) for the statics and later for the dynamics of thin-walled structures [15]. This method shows high efficiency for middle-order theories but becomes too complex if the higher order approximations are needed. A combined variational asymptotic method was developed by Berdichevskiy [7]. An alternative approach consists in the use of the power series (see [26-30]) or in the use of Legendre polynomials [33]. An efficient method of sampling surfaces is developed in [25].

A dimensional reduction formalism proposed by Vekua [34, 35] became one of the most powerful methods of construction of high-order shell theories; it is "… capable of approximating the solution of the full, three-dimensional problem… in various norms, a feature not shared by models obtained from, for example, asymptotic expansion techniques" [32]. The Vekua's approach was used for the analyze of thick isotropic [35] or anisotropic [20] and layered [31] shell structures, in statics [20, 32, 34, 35] and dynamics [16, 18]; it was also efficiently combined with FEM [32]. An extension of the Vekua's theory to arbitrarily thick non-shallow shells was performed by Amosov [1-3]. The improvement of the dimensional reduction formalism based on the biorthogonal expansions [5] and allowing one the use of a wide range of base systems was proposed in [36, 37]. The further development of the reductional method of construction of the shell theory hierarchy may consist in the use of approaches of the analytical mechanics of continua [23]. The generalized Lagrange equations of the second kind were constructed in [38] for anisotropic shells of constant thickness and in [39, 42-44] for shells of variable thickness. These equations were used for the analyze of normal waves in thin-walled systems [40-44]. The proposed approach seems to be quite efficient; it must be noted nevertheless that the "simplified" or "elementary" theory of Vekua type formulated in [38-44] does not allow the exact satisfaction of the boundary conditions on shell's faces. This disadvantage results in particular the overestimated shell stiffnesses [1, 2, 32, 35], or "thickness locking" [10]. It can be eliminated in the "extended" theory of Vekua type that uses the series residuals [17, 35] as the supplementary degrees of freedom.

Here the Lagrange formalism of analytical mechanics of continua is used to construct the extended shell theory. The shell model consists in the set of field variables, the surface Lagrangian density, and the constraint equations defined on the two-dimensional manifold corresponding to the base surface. The supplementary constraints are derived from the boundary conditions on shell's faces. This constrained problem is solved by the Lagrange multipliers method that results the two-dimensional initial-boundary value problem of the general shell theory.

2. Generalized Lagrange's equations of the second kind for constrained continuous systems

Let us consider an arbitrary continuous mechanical system in $S \subset \mathrm{R}^n$, $n \in \mathrm{N}$; $\Gamma = \partial S$. This system can be defined within the configuration space Ω, with a set of the field variables of the first kind q_I [23], the spatial density L_S and the hypersurface density L_Γ of the Lagrangian:

$$q_I = q_I\left(\xi^i, t\right), \quad I = 1\ldots N, \quad i = 1\ldots n; \quad L_S = L_S\left(q_i, \dot{q}_I, \mathrm{L}_J\left[q_I\right]\right); \quad L_\Gamma = L_\Gamma\left(q_I\right),$$

and with the constraint equations that can be represented in the following general form:

$$f^Q\left(q_I, \dot{q}_I, \mathrm{C}_P\left[q_I\right]\right) = 0, \quad Q = 1\ldots M_F \in \mathrm{N}, \quad P = 1\ldots M_C \in \mathrm{N}. \tag{1}$$

Here ξ^i are the Lagrange coordinate and t is the time variable; $\mathrm{L}_J, \mathrm{C}_P$ are linear operators; Einstein rule of summation is used here and below for both Latin and Greek indices.

Let us define spatial and hypersurface scalar products:

$$\left(u, v\right)_S = \int\limits_S uv \, dS; \quad \left(u, v\right)_\Gamma = \int\limits_\Gamma uv \, d\Gamma. \tag{2}$$

Therefore taking into account (1, 2) and using the Lagrange multipliers method we can derive the Lagrange equations of the second kind and the natural boundary conditions in the following form:

$$\begin{gathered} -\frac{\partial}{\partial t}\left(\frac{\partial L_S}{\partial \dot{q}_I} + \lambda_Q \frac{\partial f^Q}{\partial \dot{q}_I}\right) + \frac{\partial L_S}{\partial q_I} + \lambda_Q \frac{\partial f^Q}{\partial q_I} + \mathrm{L}_J^*\left[\frac{\partial L_S}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}\right] + \mathrm{C}_P^*\left[\lambda_Q \frac{\partial f^Q}{\partial\left(\mathrm{C}_P\left[q_I\right]\right)}\right] = 0; \\ \left\{\frac{\partial L_\Gamma}{\partial q_I} + \mathrm{B}_J^L\left[\frac{\partial L_S}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}\right] + \mathrm{B}_P^C\left[\lambda_Q \frac{\partial f^Q}{\partial\left(\mathrm{C}_P\left[q_I\right]\right)}\right]\right\}\delta q_I\Bigg|_\Gamma = 0. \end{gathered} \tag{3}$$

Here the Lagrange multipliers are denoted as λ_Q; B_J^L, B_P^C are its boundary operators and L_J^*, C_P^* are the operators being adjoined to L_J, C_P with respect to the scalar products (2).

Let us consider below a particular case: $C_J \equiv L_J$. Therefore the dynamic equations and its boundary conditions (3) can be written in a following short notation:

$$\partial_t P^I = \mathrm{L}_J^* \tilde{T}^{IJ} + Q^I; \qquad \left\{\bar{Q}^I + B_J \tilde{T}^{IJ}\right\}\delta q_I\Big|_\Gamma = 0. \tag{4}$$

Here the new generalized forces $\tilde{T}^{IJ}$ and $\tilde{Q}^I$ accounting the effect of the constraint are introduced, and the following relationships can be treated as generalized constitutive equations:

$$\begin{gathered} \tilde{T}^{IJ} = T^{IJ} + R^{IJ}; \quad T^{IJ} = \frac{\partial L_S}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}; \quad R^{IJ} = \lambda^Q \frac{\partial f_Q}{\partial\left(\mathrm{L}_J\left[q_I\right]\right)}; \quad P^I = \frac{\partial L_S}{\partial \dot{q}_I}; \\ \tilde{Q}^I = Q^I + R^I; \quad Q^I = \frac{\partial L_S}{\partial q_I}; \quad R^I = \lambda^Q \frac{\partial f_Q}{\partial q_I}; \quad \bar{Q}^I = \frac{\partial L_\Gamma}{\partial q_I}, \end{gathered} \tag{5}$$

where P^I are the generalized impulses corresponding to the time derivatives of the field variables q_I.

3. Construction of the extended shell theory of N$^{\text{th}}$ order

3.1. Three-dimensional formulation of the linear elasticity problem for a shell

Let us consider an anisotropic elastic shell $V \subset \mathrm{R}^3$, $\partial V = S_\pm \oplus S_B$ with smooth faces $S_\pm$ and a lateral surface S_B. In general, Lagrangian $\mathcal{L}$ for the shell can be written as follows [43, 44]:

$$\mathcal{L} = \int_{S_0} \left(T - W + A_V\right) dV + \int_{S_B} \mathbf{q}_B \cdot \mathbf{u} dS_B, \quad T = \tfrac{1}{2}\rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}; \quad W = \tfrac{1}{2}\boldsymbol{\sigma} : \mathbf{d}, \quad \mathbf{d} = \nabla \otimes \mathbf{u}; \quad A_V = \rho \mathbf{F} \cdot \mathbf{u}, \tag{6}$$

where $\mathbf{u}$ is the displacement vector, ρ is the mass density, $\boldsymbol{\sigma}$ and $\mathbf{d}$ denote the symmetric stress and distortion tensors, $\mathbf{F}$ is the resultant mass force vector, and $\mathbf{q}_B$ is the resultant force vectors on the lateral surface S_B.

The shell coordinate surface S can be defined by the vector radius $\mathbf{r}(M_0)$, $M_0 \in S_0$. Let us consider the curvilinear coordinates $\xi^\alpha \in D_\xi \subseteq \mathrm{R}^2$, $\alpha = 1,2$; the tangent fibration $T_M S$ of the two-dimensional manifold S is determined by the pair of base vectors $\mathbf{r}_\alpha(M_0)$ and the corresponding metric tensor $\mathbf{a}$:

$$\mathbf{r}_\alpha\left(M_0\right) = \partial \mathbf{r} / \partial \xi^\alpha; \quad \mathbf{a} = a_{\alpha\beta} \mathbf{r}^\alpha \otimes \mathbf{r}^\beta; \quad a_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta.$$

A point $M \in (V \setminus S)$ can be defined by the vector radius $\mathbf{R}(M_0, \xi^3)$, where ξ^3 is the normal coordinate:

$$\mathbf{R}\left(M_0, \xi^3\right) = \mathbf{r}\left(M_0\right) + \xi^3 \mathbf{n}\left(M_0\right); \quad \xi^3 \in \left[h_+, h_-\right] \subset \mathrm{R}; \quad S_\pm : \xi^3 = h_\pm; \quad \mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 / \sqrt{a}, \quad a = \det\left(a_{\alpha\beta}\right).$$

A feature of the presented reduction technique is the simultaneous use of two base systems, the main (holonomic) basis $\mathbf{R}^\alpha, \mathbf{n}$ and the non-holonomic one $\mathbf{r}^\alpha, \mathbf{n}$. The base vectors $\mathbf{R}^\alpha(M)$ are represented as follows [35]:

$$\begin{aligned}
&\mathbf{R}_\alpha\left(M_0 \zeta\right) = A_{\alpha\cdot}^{\cdot\beta}\left(M_0, \zeta\right) \mathbf{r}_\beta\left(M_0\right), \quad A_{\alpha\cdot}^{\cdot\beta} = \delta_\alpha^\beta - \left(\zeta h + \bar{h}\right) b_\alpha^\beta; \quad \zeta = \left(\xi^3 - \bar{h}\right)/h, \ 2\bar{h} = h_+ + h_-, \ 2h = h_+ - h_-; \\
&\mathbf{R}^\alpha\left(M_0, \zeta\right) = A_{\cdot\beta}^{\alpha\cdot}\left(M_0, \zeta\right) \mathbf{r}^\beta\left(M_0\right); \quad A_{\cdot\beta}^{\alpha\cdot} = \mu^{-1}\left[\delta_\beta^\alpha - \left(\zeta h + \bar{h}\right)\left(b_\lambda^\lambda \delta_\beta^\alpha - b_\alpha^\beta\right)\right]; \\
&\bar{\mu} = 1 + \bar{H} + \tilde{h}^2 \bar{K} - 2\zeta\left(\bar{H} + \tilde{h}\bar{K}\right) + \zeta^2 \bar{K}; \quad \bar{H} = \tfrac{1}{2} h b_\alpha^\alpha, \quad \bar{K} = h^2 \det\left(b_\alpha^\beta\right); \quad \tilde{h} = h^{-1}\bar{h};
\end{aligned} \tag{7}$$

Using (7) the two-point representation of the distortion and stress tensor for the shell is allowed [43, 44]:

$$\begin{aligned}
&\mathbf{d} = \bar{d}_{\alpha\beta} \mathbf{r}^\alpha \otimes \mathbf{R}^\beta + \bar{d}_{3\beta} \mathbf{n} \otimes \mathbf{R}^\beta + \bar{d}_{\alpha 3} \mathbf{r}^\alpha \otimes \mathbf{n} + \bar{d}_{33} \mathbf{n} \otimes \mathbf{n}, \\
&\bar{d}_{\alpha\beta} = \bar{\nabla}_\beta u_\alpha + h_\beta \partial_\zeta u_\alpha - b_{\beta\alpha} u_3; \quad \bar{d}_{\alpha 3} = h^{-1} \partial_\zeta u_\alpha; \quad \bar{d}_{3\beta} = \bar{\nabla}_\beta u_3 + h_\beta \partial_\zeta u_3 + b_\beta^\alpha u_\alpha; \quad \bar{d}_{33} = h^{-1} \partial_\zeta u_3; \\
&\boldsymbol{\sigma} = \bar{\mu}\left(\sigma^{\alpha\beta} \mathbf{r}_\alpha \otimes \mathbf{R}_\beta + \sigma^{\alpha 3} \mathbf{r}_\alpha \otimes \mathbf{n} + \sigma^{3\beta} \mathbf{n} \otimes \mathbf{R}_\beta + \sigma^{33} \mathbf{n} \otimes \mathbf{n}\right) = s^{\alpha\beta} \mathbf{r}_\alpha \otimes \mathbf{r}_\beta + s^{\alpha 3} \mathbf{r}_\alpha \otimes \mathbf{n} + s^{3\beta} \mathbf{n} \otimes \mathbf{r}_\beta + s^{33} \mathbf{n} \otimes \mathbf{n};
\end{aligned} \tag{8}$$

here $\bar{\nabla}$ denotes the covariant derivative on the tangent fibration $T_M S$. Taking into account (7) and (8) the Lagrangian (6) can be written in the following form [43] ready for the dimensional reduction:

$$\mathcal{L} = \int_{S_0} \left\{ \tfrac{1}{2}\left(\bar{\mu}\rho \dot{u}^i, \dot{u}_i\right)_1 - \tfrac{1}{2}\left(\bar{\mu} s^{ij}, \bar{d}_{ij}\right)_1 + \left(\bar{\mu}\rho F^i, u_i\right)_1 \right\} h dS_0 + \int_\Gamma \mu_B q_B^i u_i h d\Gamma, \quad i,j = 1 \ldots 3; \quad \left(u,v\right)_1 = \int_{-1}^{1} uv \, d\zeta. \tag{9}$$

The boundary conditions on the faces of a shell can be represented in the invariant notation as

$$\forall M \in S_\pm \quad \mathbf{s}\big|_\pm \cdot \mathbf{v} \equiv [\mathbf{C} : (\nabla \otimes \mathbf{u})]\big|_\pm = \mathbf{q}^\pm, \tag{10}$$

where $\mathbf{C}$ is the elastic constant tensor and $\mathbf{v}$ is the normal unit on the face of the shell. Accounting the geometric relationships shown in [43] we can rewrite the boundary conditions (10) with respect to the vector basis $\mathbf{r}^{\alpha}, \mathbf{n}$ that is defined on the base surface S [39, 43]:

$$\mp h^{\pm}_{\beta} s^{i\beta}\Big|_{S\pm} \pm s^{i3}\Big|_{S\pm} = \overline{q}^{i}_{\pm}, \quad i = 1\ldots 3, \quad \beta = 1\ldots 2; \quad h^{\pm}_{\beta} = \partial_{\delta} h^{\pm}. \tag{11}$$

Thus the boundary conditions are translated from the faces to the base surface and its reduction is now allowed.

3.2. Construction of a two-dimensional model of a shell

The groundwork of the presented approach consists in the application of the dimensional reduction method [35] to the Lagrangian (9) to construct the new scalar generator function defined "on the base surface", and then use it to derive the "two-dimensional" dynamic equations. Let us introduce an arbitrary biorthogonal base system in $[-1,1]$:

$$\mathrm{p}_{k)}(\zeta), \quad \mathrm{p}^{(m)}(\zeta): \quad \left(\mathrm{p}_{(k)}, \mathrm{p}^{(m)}\right)_1 = \delta^{(m)}_{(k)}.$$

Let us use the non-holonomic basis $\mathbf{r}^{\alpha}, \mathbf{n}$ being independent from the normal coordinate to determine the displacement vector. Supposing its components $u_i(M_0, \zeta)$ be square integrable over $\zeta \in [-1,1]$ we can write

$$\mathbf{u} = u^{(k)}_{\alpha} \mathrm{p}_{(k)} \mathbf{r}^{\alpha} + u^{(k)}_{3} \mathrm{p}_{(k)} \mathbf{n} = u^{\alpha}_{(k)} \mathrm{p}^{(k)} \mathbf{r}_{\alpha} + u^{(k)}_{3} \mathrm{p}_{(k)} \mathbf{n}; \quad k = 1\ldots N; \quad u^{(k)}_{i} = \left(u_i, \mathrm{p}^{(k)}\right)_1, \quad u^{i}_{(k)} = \left(u^{i}, \mathrm{p}_{(k)}\right)_1. \tag{12}$$

We can represent the components covariant components of the distortion tensor (8) by the biorthogonal expansions [5] with respect to the base system $\mathrm{p}_{(k)}(\zeta)$ on the groundwork of (12):

$$\begin{aligned} &\overline{d}^{(k)}_{\alpha\beta} = \overline{\nabla}_{\beta} u^{(k)}_{\alpha} + H^{(\bullet k)}_{\beta(m\bullet)} u^{(m)}_{\alpha} - b_{\beta\alpha} u^{(k)}_{3}; \quad \overline{d}^{(k)}_{\alpha 3} = h^{-1} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{\alpha}; \\ &\overline{d}^{(k)}_{3\beta} = \overline{\nabla}_{\beta} u^{(k)}_{3} + H^{(\bullet k)}_{\beta(m\bullet)} u^{(m)}_{3} + b^{\alpha}_{\gamma} u^{(k)}_{\alpha}; \quad \overline{d}^{(k)}_{33} = h^{-1} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{3}, \quad k, m, n \in [0, N] \cup \mathbb{Z} \end{aligned} \tag{13}$$

The notations of the linear operators are taken from the article [39]:

$$H^{(\bullet k)}_{\alpha(m\bullet)} = -D^{(\bullet k)}_{(n\bullet)} \left[\left(\partial_{\alpha} \overline{h}\right) h^{-1} \delta^{(n)}_{(m)} + \partial_{\alpha} (\ln h) Z^{(\bullet n)}_{(m\bullet)} \right]; \quad D^{(\bullet m)}_{(k\bullet)} = \left(\tfrac{d}{d\zeta} \mathrm{p}_{(k)}, \mathrm{p}^{(m)}\right)_1; Z^{(\bullet n)}_{(m\bullet)} = \left(\zeta \mathrm{p}_{(m)}, \mathrm{p}^{(n)}\right)_1. \tag{14}$$

The construction of the matrices (14) can be automated by use of any computer algebra software supporting the tensor algebra. Using (9), (12) and (13), the Lagrangian surface density can be defined on $T_M S$:

$$\begin{aligned} L_S = &\tfrac{1}{2} \rho^{(m)}_{(k)} \left(\dot{u}^{\alpha}_{(m)} \dot{u}^{(k)}_{\alpha} + \dot{u}^{3}_{(m)} \dot{u}^{(k)}_{3} \right) + P^{i}_{(k)} u^{(k)}_{i} - \tfrac{1}{2} \Big[s^{\alpha\beta}_{(k)} \left(\overline{\nabla}_{\beta} u^{(k)}_{\alpha} + H^{(\bullet k)}_{\beta(m\bullet)} u^{(m)}_{\alpha} - b_{\beta\alpha} u^{(k)}_{3} \right) + h^{-1} s^{\alpha 3}_{(k)} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{\alpha} + \\ &+ s^{3\beta}_{(k)} \left(\overline{\nabla}_{\beta} u^{(k)}_{3} + H^{(\bullet k)}_{\beta(m\bullet)} u^{(m\bullet)}_{3} + b^{\alpha}_{\gamma} u^{(k)}_{\alpha} \right) + h^{-1} s^{33}_{(k)} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{3} \Big], \quad \rho^{(m)}_{(k)} = \left(\rho h \mathrm{p}_{(k)}, \mathrm{p}^{(m)}\right)_1; \quad P^{i}_{(k)} = \left(\rho h F^{i}, \mathrm{p}_{(k)}\right)_1. \end{aligned} \tag{15}$$

After the use of (13) the boundary conditions on the faces (11) become the supplementary constraints of type (1) on $T_M S$. These constraint equations can be written as follows:

$$\begin{aligned} &\overline{\overline{C}}{}^{i\gamma\beta}_{\pm(k)} \left(\overline{\nabla}_{\delta} u^{(k)}_{\gamma} + H^{(\bullet k)}_{\delta(m\bullet)} u^{(m)}_{\gamma} - b_{\gamma\delta} u^{(k)}_{3} \right) + \overline{C}^{i3\gamma}_{\pm(k)} h^{-1} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{\gamma} + \overline{C}^{i33}_{\pm(k)} h^{-1} D^{(\bullet k)}_{(m\bullet)} u^{(m)}_{3} + \\ &+ \overline{\overline{C}}{}^{i\gamma 3}_{\pm(k)} \left(\overline{\nabla}_{\delta} u^{(k)}_{3} + H^{(\bullet k)}_{\delta(m\bullet)} u^{(m)}_{3} + b^{\gamma}_{\delta} u^{(k)}_{\gamma} \right) - \mu_{\pm} q^{i}_{\pm} = 0, \quad i = 1\ldots 3, \quad \mu_{\pm} = \mu\big|_{\zeta=\pm 1}. \end{aligned} \tag{16}$$

Here the surface values of the generalized elastic constants $\bar{\bar{C}}^{ijpq}$ [39, 42, 44] are introduced:

$$\bar{\bar{C}}^{ij\delta}_{\pm(k)} = \bar{C}^{i3j\delta}_{\pm} \mathrm{p}_{(k)}(\pm 1) + h^{\pm}_{\beta} \bar{\bar{C}}^{i\beta j\delta}_{\pm} \mathrm{p}_{(k)}(\pm 1); \quad \bar{\bar{C}}^{ijpq}_{\pm} = \bar{\bar{C}}^{\,ijpq}\Big|_{\zeta=\pm 1},$$

where the components $\bar{\bar{C}}^{ijkl}$ of the referenced to the concomitant basis $\mathbf{r}_\alpha$, $\mathbf{r}_3 \equiv \mathbf{n}$ elastic tensor are defined in [43]:

$$\bar{\bar{C}}^{i\beta j\delta} = A^{\beta\cdot}_{\cdot\lambda} A^{\delta\cdot}_{\cdot\mu} C^{i\lambda j\mu}; \quad \bar{C}^{i3j\delta} = A^{\delta\cdot}_{\cdot\mu} C^{i3j\mu}; \quad i,j = 1,2,3; \quad \delta,\gamma,\lambda,\mu = 1,2.$$

The Lagrange equations for the extended shell theory and their natural boundary conditions (4) can be represented in the following form similar to the elementary shell theory [39, 43]:

$$\begin{aligned}
&\rho^{(m)}_{(k)} \ddot{u}^{\alpha}_{(m)} = \bar{\nabla}_\beta \tilde{s}^{\alpha\beta}_{(k)} - H^{(\cdot m)}_{\beta(k\cdot)} \tilde{s}^{\alpha\beta}_{(m)} - b^{\alpha}_{\beta} \tilde{s}^{3\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \tilde{s}^{\alpha 3}_{(m)} + P^{\alpha}_{(k)};\\
&\rho^{(m)}_{(k)} \ddot{u}^{3}_{(m)} = \bar{\nabla}_\beta \tilde{s}^{3\beta}_{(k)} - H^{(\cdot m)}_{\beta(k\cdot)} \tilde{s}^{3\beta}_{(m)} + b_{\alpha\beta} \tilde{s}^{\alpha\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \tilde{s}^{33}_{(m)} + P^{3}_{(k)};\\
&\left(\tilde{s}^{i\beta}_{(k)} \nu_\beta - q^{i}_{B(k)}\right) \delta u^{(k)}_i \Big|_{M_0 \in \Gamma} = 0; \quad i = 1,2,3; \quad \beta = 1,2.
\end{aligned} \tag{17}$$

The shell model consists in (16), (17), the kinematic equations and the initial conditions presented in [43], and the constitutive equations considering the boundary conditions on the faces by means of the Lagrange multipliers $\lambda^{\pm}_i$ (5) (instead of the residual series [35]):

$$\begin{aligned}
&\tilde{s}^{ij}_{(k)} = \bar{\bar{C}}^{ij\gamma\delta}_{(km)} \bar{\nabla}_\delta u^{(m)}_\gamma + \bar{\bar{C}}^{ij3\delta}_{(km)} \bar{\nabla}_\delta u^{(m)}_3 + \bar{C}^{ij\gamma}_{(km)} u^{(m)}_\gamma + \bar{C}^{ij3}_{(km)} u^{(m)}_3 + \lambda^{\pm}_{\varepsilon} \bar{\bar{C}}^{\varepsilon ij}_{\pm(k)} + \lambda^{\pm}_{3} \bar{\bar{C}}^{3ij}_{\pm(k)}; \quad \bar{\bar{C}}^{ij\gamma\delta}_{(km)} = \left(\bar{\mu} \bar{\bar{C}}^{ij\gamma\delta} \mathrm{p}_{(k)}, \mathrm{p}_{(m)}\right)_1;\\
&\bar{C}^{ij\gamma}_{(km)} = H^{(\bullet n)}_{\delta(k\bullet)} \bar{\bar{C}}^{\,ij\gamma\delta}_{(nm)} + b^{\gamma}_{\delta} \bar{\bar{C}}^{i\beta 3\delta}_{(km)} + h^{-1} D^{(\bullet n)}_{(k\bullet)} \bar{C}^{ij\gamma 3}_{(nm)}; \quad \bar{C}^{ij3}_{(km)} = H^{(\bullet n)}_{\delta(k\bullet)} \bar{C}^{ij3\delta}_{(nm)} - b_{\gamma\delta} \bar{C}^{ij\gamma\delta}_{(km)} + h^{-1} D^{(\bullet n)}_{(k\bullet)} C^{ij33}_{(nm)}.
\end{aligned} \tag{18}$$

The equations (18) account the effect of the resultant force vector fields $\mathbf{q}_{\pm}$ on the shell faces on the generalized forces of the constructed two-dimensional shell model.

4. Conclusions

A new type of thick anisotropic shells theory is constructed on the basis of the Lagrangian analytical mechanics of continua, the dimensional reduction approach, and the biorthogonal expansion method. The proposed variational model consists in the configurational space with the set of field variables of the first kind, the Lagrangian density, and the supplementary constraints for the field variables defined on the two-dimensional manifold corresponding to the shell base surface. Contrarily to the earlier variants of this theory [39-44] the boundary conditions on shells' faces are satisfied; the constraint equations are derived from these boundary conditions translated to the base surface of a shell. The use of the Lagrange multipliers method for the constrained variational problem allows one to obtain the dynamic equations similar to the ones of the simplified shell theory [43]. The constitutive equations for the new generalized forces contain the Lagrange multipliers, therefore the generalized stiffnesses are determined accounting the resultant force vector on the faces.

The proposed approach allows one to use the advantages of the variational formalism [23] for construction of extended shell theories of Vekua type [35] together with the advantages of the biorthogonal expansions [5].

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